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LETTER TO THE EDITOR

On a medium constraint arising directly from Maxwell's equations

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Abstract. The mathematical structure of Maxwell's equations provides a constraint on the constitutive parameters of linear homogeneous media. This is exemplified by showing that the non-reciprocity parameter of bi-isotropic media is superfluous, confirming independent results (based on covariance and uniqueness arguments) that non-reciprocal bi-isotropic media do not exist.

Artifically tailored, complex materials are currently widely studied by materials scientists because they offer exciting prospects for technological applications from the microwave to the optical regimes. These experimental activities have spawned a wave of new interest at the theoretical level to gain a better understanding of electromagnetic wave propagation, radiation and scattering in complex media. In electromagnetics, the most general linear material is the so-called bi-anisotropic medium. Such a substance is characterized by a large number of parameters (36 complex scalars, to be exact), which, ideally, would follow from a quantum-statistical description of bulk matter. Needless to say, such models are still in their infancy. Whatever is available is based on more or less sophisticated effective medium theories such as long-wavelength scattering models [1].

There has been a longstanding tradition to provide constraints on medium parameters, i.e. relations that reduce the number of degrees of freedom of a certain type of material as a consequence of certain requirements. Significantly, such requirements are generally derived from *physical* conditions [2, 3], losslessness and reciprocity being just two examples (reciprocity is the symmetry of action and reaction in a given medium).

Much less attention has been paid to constraints arising from the structure of the *governing differential equations (Maxwell's equations) themselves.*

To investigate the mathematical structure of Maxwell's equations with a view to a parameter constraint provides the motivation for this letter. It consists of a simple and straightforward exercise and will be pursued here for a general bi-isotropic medium. Reciprocal bi-isotropic media, more commonly known as chiral media, are found abundantly in nature and can also be manufactured in the form of synthetic particulate composites [4–7].

While a chiral medium is characterized by three medium parameters, a general *non-reciprocal bi-isotropic medium (the most general medium with direction-independent* parameters) requires four scalars for its description. However, the great interest in the latter [7, 8] during recent years is doubly surprising: first, mathematical generalizations of the field solutions for chiral media, which are usually available in closed form, to the non-reciprocal bi-isotropic case are mostly trivial; and second, and more importantly, there is currently

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no experimental evidence that non-reciprocal bi-isotropic media exist in either natural or manufactured forms [9, 10].

The purpose of this letter is therefore twofold. On the one hand, we want to draw attention to mathematical structures hidden within Maxwell's equations which lead directly to a parameter constraint and thus limit the admissible media. On the other hand, by choosing a general bi-isotropic medium for this exercise, the repercussions of this parameter constraint will shed some new light on the concept of reciprocity in the context of bi-isotropic media.

Maxwell's equations for frequency-dependent fields are given by

$$\nabla \times \boldsymbol{H}(\boldsymbol{x}, \omega) + \mathrm{i}\omega \boldsymbol{D}(\boldsymbol{x}, \omega) = \boldsymbol{J}(\boldsymbol{x}, \omega) \tag{1}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{x}, \omega) - \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{x}, \omega) = 0 \tag{2}$$

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{x}, \omega) = \rho(\boldsymbol{x}, \omega) \tag{3}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, \omega) = 0. \tag{4}$$

The electromagnetic field phasors $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ and $\boldsymbol{B}$ depend on the spatial coordinate vector $\boldsymbol{x}$ and the circular frequency ω, an $\exp(-\mathrm{i}\omega t)$ time-dependence is suppressed throughout (the frequency domain is chosen simply for convenience; conversion of the results obtained in this letter to the time domain is straightforward). It is apparent that (3) and (4) are not independent of the curl equations (1) and (2) because, implicitly, the electric charge density ρ and the electric current density $\boldsymbol{J}$ fulfil the continuity equation

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{x}, \omega) - \mathrm{i}\omega\rho(\boldsymbol{x}, \omega) = 0. \tag{5}$$

Therefore, it becomes clear that (1)–(4) do not form a self-consistent system of differential equations for the complete determination of the fields $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ and $\boldsymbol{B}$. Two more equations need to be added (and, if the medium is finite, appropriate boundary conditions must be formulated). These are commonly known as constitutive relations because they contain information about the material properties of the medium under consideration.

The constitutive relations of general homogeneous bi-isotropic media in the Boys–Post form [11] can be stated as

$$\boldsymbol{D}(\boldsymbol{x}, \omega) = \epsilon(\omega)\boldsymbol{E}(\boldsymbol{x}, \omega) + (\alpha(\omega) + \beta(\omega))\boldsymbol{B}(\boldsymbol{x}, \omega) \tag{6}$$

$$\boldsymbol{H}(\boldsymbol{x}, \omega) = (-\alpha(\omega) + \beta(\omega))\boldsymbol{E}(\boldsymbol{x}, \omega) + \boldsymbol{B}(\boldsymbol{x}, \omega)/\mu(\omega) \tag{7}$$

where, as is common in modern physics literature, $\boldsymbol{E}$ and $\boldsymbol{B}$ are taken to be the primitive fields whereas $\boldsymbol{D}$ and $\boldsymbol{H}$ are the induction fields. Four frequency-dependent parameters describe a homogeneous general bi-isotropic medium: permittivity ϵ, permeability μ, chirality parameter β and non-reciprocity parameter α (all functions of frequency ω). The widely available chiral (i.e. reciprocal bi-isotropic) media arise when the specialization $\alpha(\omega) = 0$ is made [12].

The stage is now set to eliminate the induction fields $\boldsymbol{D}$ and $\boldsymbol{H}$ from Maxwell's equations (1) and (3) by using (6) and (7). One finds

$$\begin{aligned} &\nabla \times \boldsymbol{B}(\boldsymbol{x}, \omega)/\mu(\omega) + \mathrm{i}\omega\epsilon(\omega)\boldsymbol{E}(\boldsymbol{x}, \omega) + 2\mathrm{i}\omega\beta \boldsymbol{B}(\boldsymbol{x}, \omega) \\ &\qquad + (-\alpha(\omega) + \beta(\omega))[\nabla \times \boldsymbol{E}(\boldsymbol{x}, \omega) - \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{x}, \omega)] = \boldsymbol{J}(\boldsymbol{x}, \omega) \end{aligned} \tag{8}$$

$$\epsilon(\omega)\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, \omega) + (\alpha(\omega) + \beta(\omega))[\nabla \cdot \boldsymbol{B}(\boldsymbol{x}, \omega)] = \rho(\boldsymbol{x}, \omega). \tag{9}$$

By using the two differential equations (2) and (4), which only contain the primitive fields, the terms which were grouped in square brackets vanish identically. Then, the following self-consistent system of partial differential equations for the primitive fields $\boldsymbol{E}(\boldsymbol{x},\omega)$, $\boldsymbol{B}(\boldsymbol{x},\omega)$ arises:

$$\nabla\times\boldsymbol{B}(\boldsymbol{x},\omega)/\mu(\omega)+\mathrm{i}\omega\epsilon(\omega)\boldsymbol{E}(\boldsymbol{x},\omega)+2\mathrm{i}\omega\beta\boldsymbol{B}(\boldsymbol{x},\omega)=\boldsymbol{J}(\boldsymbol{x},\omega) \tag{10}$$

$$\nabla\times\boldsymbol{E}(\boldsymbol{x},\omega)-\mathrm{i}\omega\boldsymbol{B}(\boldsymbol{x},\omega)=0 \tag{11}$$

$$\epsilon(\omega)\nabla\cdot\boldsymbol{E}(\boldsymbol{x},\omega)=\rho(\boldsymbol{x},\omega) \tag{12}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{x},\omega)=0. \tag{13}$$

The key observation relating to (10)–(13) is that the non-reciprocity parameter α does not occur in them. This independence of the governing differential equations of the non-reciprocity parameter α means that instead of employing (6) and (7) one could equally well have used the constitutive relations

$$\boldsymbol{D}(\boldsymbol{x},\omega)=\epsilon(\omega)\boldsymbol{E}(\boldsymbol{x},\omega)+\beta(\omega)\boldsymbol{B}(\boldsymbol{x},\omega) \tag{14}$$

$$\boldsymbol{H}(\boldsymbol{x},\omega)=\beta(\omega)\boldsymbol{E}(\boldsymbol{x},\omega)+\boldsymbol{B}(\boldsymbol{x},\omega)/\mu(\omega) \tag{15}$$

i.e. those of a chiral medium, to arrive at (10)–(13).

The mathematical manipulations above, simple as they are, nevertheless lead to an important result. When the governing differential equations are written in terms of the primitive fields only, they are seen to be insensitive to the fourth parameter (i.e. the non-reciprocity parameter α) *of a general bi-isotropic medium. Now,* $\boldsymbol{E}$ *and* $\boldsymbol{B}$ *are the phasors* of the primitive fields, i.e. of the physically relevant fields. Other quantities, such as power flux, radiation loss etc, can be derived directly from them.

Throughout history, physical theories have been motivated by a quest for simplicity. Guided by Occam's razor or the principle of parsimony, this quest is an attempt to minimize the number of input parameters. Consequently, the foregoing analysis leads to the inevitable conclusion that the non-reciprocity parameter α is unnecessary in the description of bi-isotropic media. It is therefore prudent to set

$$\alpha(\omega)=0 \qquad \text{for all } \omega. \tag{16}$$

Thus, in the context of bi-isotropic media, non-reciprocity is a superfluous concept. Bi-isotropic media are always reciprocal, i.e. they are necessarily chiral.

Bi-isotropic media fall into the class of linear magnetoelectric materials which are inherently bi-anisotropic. A considerable amount of experimental work, predominantly carried out in the 1960s, was mostly aimed at establishing the material properties of such magnetoelectric substances as chromium sesquioxide (Cr_2O_3). These studies were not primarily intended to address the question of the existence or non-existence of a non-reciprocal bi-isotropic medium, yet a detailed investigation of the experimental literature, conducted recently by Lakhtakia [9] (see also for detailed references to experimental studies), has comprehensively shown that experimental work to date has failed in establishing the existence of a non-reciprocal bi-isotropic medium. Furthermore, the so-called Tellegen medium [13], often used as an example for a non-reciprocal bi-isotropic medium, has never been synthesized. Indeed, closer inspection [10] shows that the Tellegen medium is not distinguishable from a simple dielectric–magnetic medium in any performable experiment, and is therefore pathological.

The main emphasis here is to show how a constraint on medium parameters can arise directly from the differential equations. In the case of bi-isotropic media, the constraint turns out to be equivalent to a requirement for reciprocity. The present result is vindicated by investigations of the most general linear homogeneous media. Bi-anisotropic materials are characterized by the constitutive relations

$$D(x,\omega) = \underline{\underline{\epsilon}}(\omega) \cdot E(x,\omega) + (\underline{\underline{\alpha}}(\omega) + \underline{\underline{\beta}}(\omega)) \cdot B(x,\omega) \tag{17}$$

$$H(x,\omega) = (-\underline{\underline{\alpha}}(\omega) + \underline{\underline{\beta}}(\omega)) \cdot E(x,\omega) + \underline{\underline{\mu}}^{-1}(\omega) \cdot B(x,\omega) \tag{18}$$

where $\underline{\underline{\epsilon}}, \underline{\underline{\mu}}, \underline{\underline{\alpha}}, \underline{\underline{\beta}}$ are now dyadics (3×3 Cartesian tensors). It was established independently, through requirements for general covariance [14, 15] and uniqueness [16], that a homogeneous bi-anisotropic medium must obey the uniformity constraint

$$\mathrm{Trace}(\underline{\underline{\alpha}}(\omega)) = 0 \qquad \text{for all } \omega. \tag{19}$$

It is important to note that constraint (19) does not imply reciprocity for a bi-anisotropic medium, it simply provides one algebraic relation for the 36 medium parameters to fulfil; yet on application to the special bi-isotropic case, requirement (16) is recovered from relation (19).

Quite clearly, constraint (16) has been derived as a consequence of the mathematical structure of Maxwell's equations, and not because of any considerations of the physical nature of the bi-isotropic medium.

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